A Measure of Directional Outlyingness with Applications to Image Data and Video

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Outline

Introduction

Two main approaches for outliers:

1. Apply outlier detection method to data set and remove outliers before analysis
2. Use a data analysis method that is robust to outliers

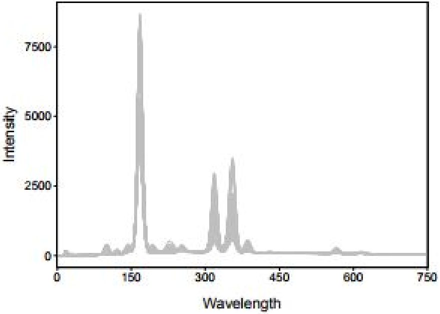
Rousseeuw et al. propose a measure of directional outlyingness to identify outliers in univariate and multivariate functional data

Motivating data

Univariate functional data:

$Y_i(d)$ are spectra at $d = 1, \ldots, 750$ wavelengths from $n = 180$ archaeological glass samples

Goal: Detect outlying curves and identify how/where they are outlying



Motivating data

Multivariate functional data:

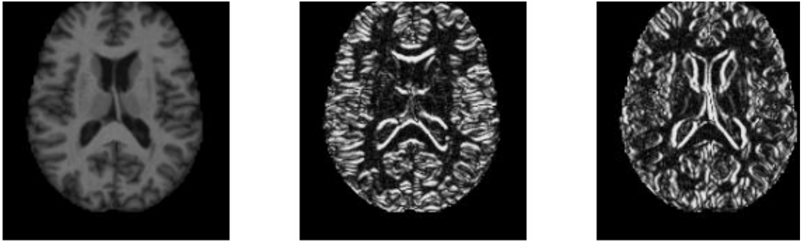


Figure: MRI image from a single patient (left) and its derivatives, horizontal (center) and vertical (right).

Motivating data

Multivariate data:

$\boldsymbol{Y}_i(j,k)$ is a trivariate response:

- j and k reference the pixel $j = 1, \ldots, 176$ and $k = 1, \ldots, 208$
- $\boldsymbol{Y}_i(j,k)$ = (grayscale value, horizontal derivative, vertical derivative) for pixel j, k
- Sample of $n = 416$ subjects

Goal: Detect outlying images and identify outlying segments of images

Key contributions

- Propose outlier detection method for univariate or multivariate functional data
- Can consider functional data with a multivariate domain (ex: images, videos)
- Method robust to skewness in data
- Computationally efficient - $\mathcal{O}(n)$ per direction

Outline

Stahel-Donoho outlyingness (SDO)

For univariate data let $Y = \{y_1, \ldots, y_n\}$

$$SDO(y; Y) = \frac{|y - med(Y)|}{MAD(Y)}$$

where *MAD* is the median average deviation:

$$MAD(Y) = \frac{med_i(|y_i - med(Y)|)}{\Phi^{-1}(0.75)}$$

Intuition: Center and scale each response using robust methods
Limitation: If “inliers” have a skewed distribution using MAD will not account for asymmetry

Directional outlyingness (DO)

Rousseeuw et al. propose directional outlyingess (DO):

- Split sample into two subsamples: those above (a) and below (b) the median
- Robustly estimate scale parameter for two subsamples
- Will account for potential skewness in data

$$DO(y;Y) = \begin{cases} \frac{y - med(Y)}{S_a(Y)} & \text{if } y \geq med(Y) \\ \frac{med(Y) - y}{S_b(Y)} & \text{if } y \leq med(Y) \end{cases}$$

$S_a(Y)$ and $S_b(Y)$ are M-estimates of the scale parameters for the two subsamples

Properties of DO

- Affine invariant:
  $DO(-y; -Y) = DO(y; Y)$ OR
  $DO(cy; cY) = DO(y, Y)$ for $c \neq 0$
- Robust to small contaminations: small deviations from parametric distributions

Outline

Multivariate DO

Let $\boldsymbol{y}$ be a d-variate point from a sample $\boldsymbol{Y} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$
The multivariate DO is:

$$DO(\boldsymbol{y}; \boldsymbol{Y}) = \sup_{\boldsymbol{v} \in \mathbb{R}^d} DO(\boldsymbol{y}^T \boldsymbol{v}; \boldsymbol{Y}^T \boldsymbol{v})$$

Intuition:

- Use projections to yield a univariate measure of outlyingness
- If a point is outlying with respect to the data set it will stand out in at least one direction

Multivariate DO in practice

- Impossible to calculate $DO(\boldsymbol{y}^T\boldsymbol{v}; \boldsymbol{Y}^T\boldsymbol{v})$ for all directions v in a d-dimensional space
- Solution:
  1. Randomly select d data points
  2. Compute hyperplane passing through the points
  3. Take direction $\boldsymbol{v}$ orthogonal to it
  4. Repeat Steps 1-3 $250d$ times to yield set of directions
  5. Compute multivariate DO for each i

Multivariate DO is also affine invariant:
It doesn't change when adding constant vector to data or multiplying by non-singular $d \times d$ matrix

Functional directional outlyingness (fDO)

Let $\boldsymbol{Y} = \{Y_1, \ldots, Y_n\}$ where each Y_i is a function on a discrete set of time points $\{t_1, \ldots, t_T\}$
The **functional directional outlyingness (fDO)** is:

$$\text{f}DO(Y_i : \boldsymbol{Y}) = \sum_{j=1}^{T} DO(Y_i(t_j); \boldsymbol{Y}(t_j))W(t_j)$$

where $W(\cdot)$ is a weight function where $\sum_{j=1}^{T} W(t_j) = 1$

- We can weight the importance of different time points
- If the we have multivariate functional data we use the Multivariate DO based on projections

fDO of glass data

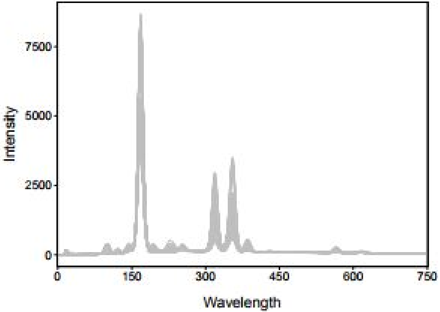


Figure: $n = 180$ glass functions.

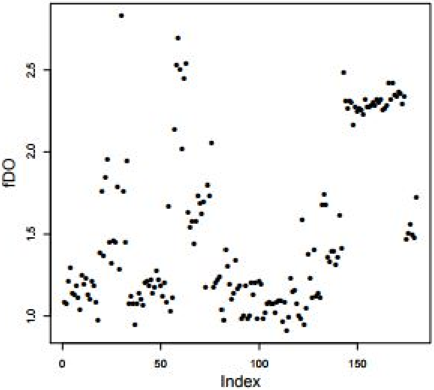
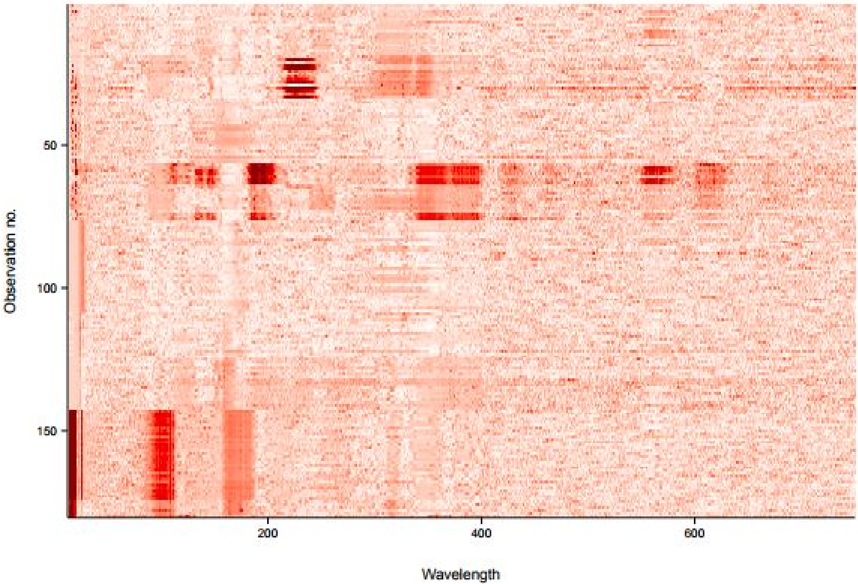


Figure: fDO of $n = 180$ glass functions.

DO of glass data



Outline

Cutoff for DO or fDO

- The DO_i and fDO_i will have a right skewed distribution
- Take a log-transformation to yield a more Gaussian distribution

$$\{LDO_1, \ldots, LDO_n\} = \{log(DO_1 + 0.1), \ldots, log(DO_n + 0.1)\}$$

- Then robustly center and normalize
- Flag $\boldsymbol{y}_i$ as an outlier if

$$\frac{LDO_i - med(LDO)}{MAD(LDO)} > \Phi^{-1}(0.995)$$

- Same process can be applied to the fDO_i

Functional outlier map (FOM)

- Extend functional outlier map of Hubert et al. 2015 to DO and propose cutoff
- The fDO for curve Y_i could be considered its 'average outlyingness'
- We need a measure for the variability of its outlyingness

$$\mathrm{v}DO(Y_i;\boldsymbol{Y}) = \frac{SD_j(DO(Y_i(t_j),\boldsymbol{Y}(t_j)))}{1+\mathrm{f}DO(Y_i;\boldsymbol{Y})}$$

- They divide by $1+\mathrm{f}DO(Y_i;\boldsymbol{Y})$ to yield relative variability
- The FOM is a scatter plot of points

$$(\mathrm{f}DO(Y_i;\boldsymbol{Y}),\mathrm{v}DO(Y_i;\boldsymbol{Y})) \quad i=1,\ldots,n$$

Cutoff for FOM

Define the combined functional outlyingess (CFO) of Y_i as:

$$CFO_i = CFO(Y_i : \boldsymbol{Y}) = \sqrt{(\mathrm{f}DO_i/med(\mathrm{f}DO))^2 + (\mathrm{v}DO_i/med(\mathrm{v}DO))^2}$$

- CFO quantifies the distance of FOM points from the origin
- Large values of CFO_i correspond to outliers
- The CFO_i will also be right skewed so again take the log-transformation to get a cutoff
- Y_i is an outlier if:

$$\frac{LCFO_i - med(LCFO)}{MAD(LCFO)} > \Phi^{-1}(0.995)$$

FOM for glass data

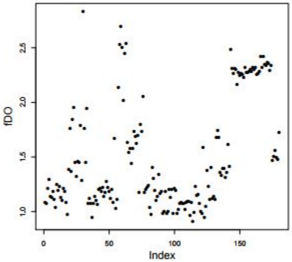


Figure: fDO of $n = 180$ glass functions.

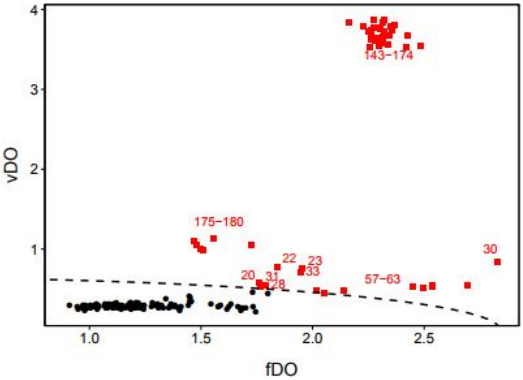


Figure: FOM of $n = 180$ glass functions.

FOM interpretation

Location of FOM points can indicate the type of outlier:

- Lower left points correspond to most central curves
- Lower right points correspond to high fDO but low variability in DO values
    - Correspond to *shift outliers*
    - Same shape but shifted on domain
- Upper left points have low fDO but high vDO
    - Correspond to *local outliers*
    - Only outlying on small part of domain
- Upper right points have high fDO and high vDO
    - Strong outliers on a large portion of the domain

Outline

Image data application

- $\boldsymbol{Y}_i(j,k)$ = (grayscale value, horizontal derivative, vertical derivative) for pixel j,k
- $j = 1,\ldots,176$, $k = 1,\ldots,208$, and $n = 1,\ldots,416$

The fDO of MRI image $\boldsymbol{Y}_i$ is:

$$\mathrm{f}DO(\boldsymbol{Y}_i;\boldsymbol{Y}) = \frac{1}{176\times 208}\sum_{j=1}^{176}\sum_{k=1}^{208} DO(\boldsymbol{Y}_i(j,k);\boldsymbol{Y}(j,k))W_{jk}$$

- $DO(\boldsymbol{Y}_i(j,k);\boldsymbol{Y}(j,k)) = \sup_{\boldsymbol{v}\in\mathbb{R}^3}\ DO(\boldsymbol{Y}_i^T\boldsymbol{v};\boldsymbol{Y}^T\boldsymbol{v})$
- $W_{jk} = 0$ if pixels not part of the brain

Image data application

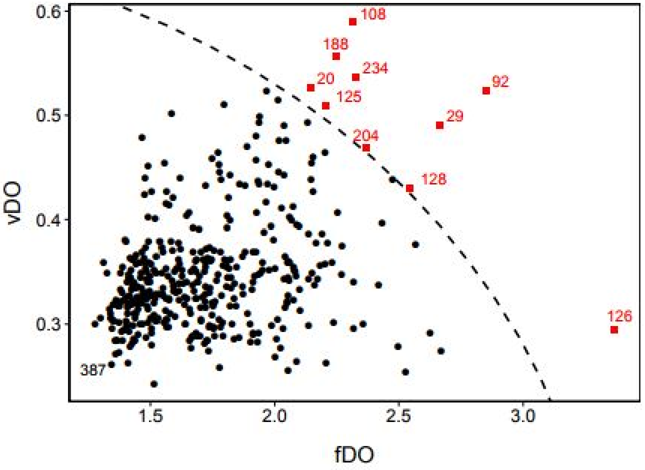
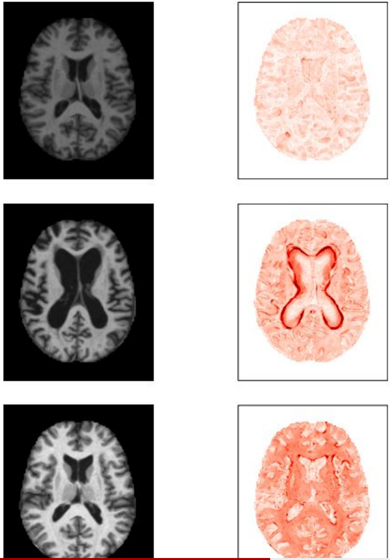


Image data application



- Subject 387 (top) has lowest CFO and is most 'central' image in dataset
- Subject 92 (middle) has high fDO and vDO - large outlying regions
- Subject 126 (bottom) has high fDO but low vDO - shifted outlier

Summary

- Propose an outlier detection method for univariate or multivariate functional data
- Can accommodate potentially skewed data
- Simulation studies show their method is more powerful than competing Adjusted Outlyingness (AO) Brys et al. (2005)

Limitations:

- Method may not account for/exploit correlation within curves
- Dai & Genton (2017) demonstrate cases where the functional outlier map has poor detection power

Competing Method: Dai & Genton (2017)

- Dai & Genton propose a similar method and graphical tool
- They use multivariate SDO as measure of directional outlyingness:

$$SDO(\boldsymbol{Y}_i(t); \boldsymbol{Y}(t)) = sup_{\|\boldsymbol{u}\|=1} \frac{\|\boldsymbol{u}^T \boldsymbol{Y}_i(t) - med(\boldsymbol{u}^T \boldsymbol{Y}(t))\|}{MAD(\boldsymbol{u}^T \boldsymbol{Y}(t))}$$

- Their definition of directional outlyingness is

$$\boldsymbol{O}(\boldsymbol{Y}_i(t); \boldsymbol{Y}(t)) = SDO(\boldsymbol{Y}_i(t); \boldsymbol{Y}(t)) \cdot \boldsymbol{v}_i(t)$$

where $\boldsymbol{v}_i(t) = \frac{\boldsymbol{Y}_i(t) - \boldsymbol{Z}(t)}{\|\boldsymbol{Y}_i(t) - \boldsymbol{Z}(t)\|}$ and $\boldsymbol{Z}(t)$ is the unique median as defined by the SDO

Competing Method: Dai & Genton (2017)

Mean directional outlyingness:

$$\boldsymbol{MO}(\boldsymbol{Y}_i;\boldsymbol{Y}) = \int_{\mathcal{I}} \boldsymbol{O}(\boldsymbol{Y}_i(t);\boldsymbol{Y}(t))w(t)dt$$

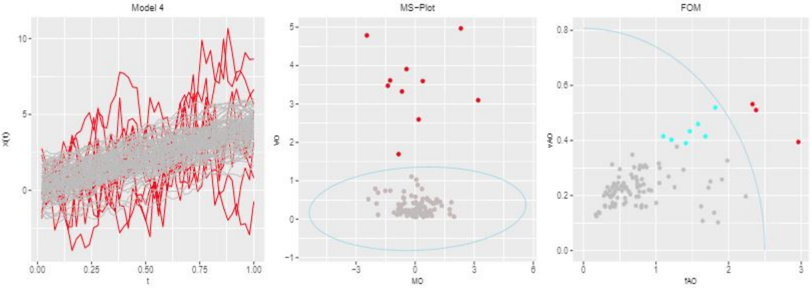
Variation of directional outlyingess:

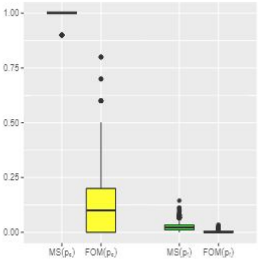
$$\boldsymbol{VO}(\boldsymbol{Y}_i;\boldsymbol{Y}) = \int_{\mathcal{I}} \|\boldsymbol{O}(\boldsymbol{Y}_i(t);\boldsymbol{Y}(t)) - \boldsymbol{MO}(\boldsymbol{Y}_i;\boldsymbol{Y})\|^2 w(t)dt$$

They introduce the magnitude-shape plot which plots:

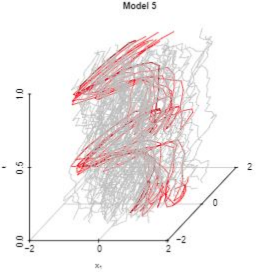
$$(\boldsymbol{MO}^T, VO)^T$$

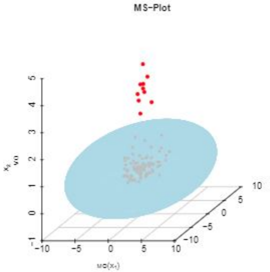
Simulations - Univariate example

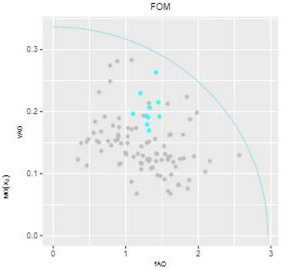


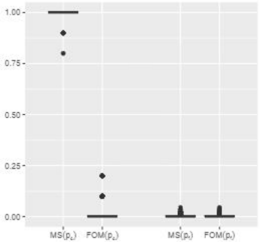


Simulations - Multivariate example









Appendix: Robust scale estimates

For univariate data $y_1 \leq y_2 \leq \cdots \leq y_n$ define $Y_a = \{y_{h+1}, \ldots, y_n\}$ and $Y_b = \{y_1, \ldots, y_h\}$ where $h = \left\lfloor \frac{n+1}{2} \right\rfloor$

$Z_a = Y_a - med(Y)$ and $Z_b = med(Y) - Y_b$

They compute initial scale estimates:

$$s_{o,a}(Y) = med(Z_b)/\Phi^{-1}(0.75) \qquad s_{o,b}(Y) = med(Z_b)/\Phi^{-1}(0.75)$$

Appendix: Robust scale estimates

The M-estimates are then:

$$s_a(Y) = s_{o,a}(Y)\sqrt{\frac{1}{2\alpha h}\sum_{z_i \in Z_a} \rho_c\left(\frac{z_i}{s_{o,a}(Y)}\right)}$$

$$s_b(Y) = s_{o,b}(Y)\sqrt{\frac{1}{2\alpha h}\sum_{z_i \in Z_b} \rho_c\left(\frac{z_i}{s_{o,b}(Y)}\right)}$$

where

- ρ_c is the Huber rho function for scale $\rho_c(t) = (\frac{t}{c})^2 I_{[-c,c]} + I_{(-\infty,c)\cup(c,\infty)}$
- c is a tuning parameter regulating bias/efficiency trade-off
- $\alpha = \int_0^\infty \rho_c(x)d\Phi(x)$